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MATHEMATICAL MONTHLY.

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PRIZE PROBLEMS FOR STUDENTS.

I. If AB, CD be chords of a circle at right angles to each other, prove that the sum of the arcs AC, BD is equal to half the circumference.

II. Prove that the cube of any number and the number itself, being divided by 6, leave the same remainder.

III. It is required to divide a given right cone into two parts, having the ratio of m to n, by a plane making an angle of p degrees with the axis. — Communicated by Prof. Wray Beattie, Iowa Wesleyan University.

IV. In any spherical triangle,

$$\frac{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}{\cos S \cos (S-A) \cos (S-B) \cos (S-C)} = \frac{\cos a \cos b \cos c - 1}{\cos a \cos b \cos c + 1},$$

in which $s = \frac{1}{2}(a+b+c)$, $S = \frac{1}{2}(A+B+C)$. — Communicated by Prof. H. H. White, Harrodsburg, Ky.

V. Describe an ellipse having its foci at any points on one of the asymptotes of an equilateral hyperbola. The ellipse will cut each branch of the hyperbola and its conjugate in two real or imaginary points. Prove that a chord joining one pair of these points will be perpendicular to one of the other similar chords. Prove it first for a circle and then for an ellipse.

Solutions of these problems must be received by November 1, 1860.

NOTES AND QUERIES.

1. *Commensurable Sides of Right-angled Triangles nearly Isosceles.* — If in the equation,

$$(2\,a_n\,b_n)^2 + (a_n^2 - b_n^2)^2 = (a_n^2 + b_n^2)^2,$$

we put $a_1 = 1,\ b_1 = 0$, then $0^2 + 1^2 = 1^2$;
$a_2 = 2,\ b_2 = 1$, " $4^2 + 3^2 = 5^2$;
$a_3 = 5,\ b_3 = 2$, " $20^2 + 21^2 = 29^2$;
$a_4 = 12,\ b_4 = 5$, " $120^2 + 121^2 = 169^2$;

and in general, if $a_n = 2\,a_{n-1} + b_{n-1}$, $b_n = a_{n-1}$, then,

$$(4\,a_{n-1}^2 + 2\,a_{n-1}\,b_{n-1})^2 + (3\,a_{n-1}^2 + 4\,a_{n-1}\,b_{n-1} + b_{n-1}^2)^2 = (5\,a_{n-1}^2 + 4\,a_{n-1}\,b_{n-1} + b_{n-1}^2)^2.$$

Also,

$$(2\,a_n\,b_n) - (a_n^2 - b_n^2) = (a_{n-1}^2 - b_{n-1}^2) - (2\,a_{n-1}\,b_{n-1}).$$
$$= (2\,a_{n-2}\,b_{n-2}) - (a_{n-2}^2 - b_{n-2}^2) = \ldots = \pm 1.$$

Additional values of a_n, b_n give

$$696^2 + 697^2 = 985^2$$
$$4060^2 + 4059^2 = 5741^2$$
$$23660^2 + 23661^2 = 33461^2$$
$$159140520^2 + 159140519^2 = 225058681^2,$$

&c. &c. &c.

— Lucius Brown, Fall River, Mass.

2. *Note on the Equation*

$$(a^2 + b^2)^2 = (a^2 - b^2)^2 + (2\,a\,b)^2.$$

The general problem is to find two square numbers whose sum or difference shall also be a square number; and it may be solved by the Diophantine Analysis.

Let x be the greater, and y the lesser number; then,

$$(1) \qquad x^2 - y^2 = \square = (x - ny)^2 = x^2 - 2\,nxy + y^2,$$

in which n is any arbitrary entire or fractional number.

From (1) we have

$$x = \frac{n^2 + 1}{2n} y.$$

We may now put y equal n, $2n$, or $3n$, &c.; but to make the value of x as simple as possible, let $y = 2n$; then $x = n^2 + 1$ and $\sqrt{(x^2 - y^2)} = n^2 - 1$.

Now, by substituting any numbers for n in these equations, we shall obtain three numbers, satisfying the conditions of the problem. Suppose $n = 4$; then $17^2 = 8^2 + 15^2$; and if in the equation

$$(n^2 + 1)^2 = (n^2 - 1)^2 + 4n^2$$

we put $n = \frac{a}{b}$, it becomes

$$\left(\frac{a^2 + b^2}{b^2}\right)^2 = \left(\frac{a^2 - b^2}{b^2}\right)^2 + \frac{4a^2}{b^2},$$

or,

$$(a^2 + b^2)^2 = (a^2 - b^2)^2 + 4a^2 b^2.$$

Hence, by assuming any two numbers for a and b, we shall find that $a^2 + b^2$ is the hypothenuse, and $a^2 - b^2$, $2ab$, the sides of a right-angled triangle. — Prof. D. Wood, University of Michigan, Ann Arbor.

3. *On the Logarithmic Solution of Cubic Equations.* — In the December number of the Monthly, as in various other publications, attention has been called to the trigonometric or logarithmic solution of equations. The subject appears of sufficient interest to merit further attention; and it is here proposed to combine the methods in a more systematic form, with such additional precepts as will adapt the operation even to persons who have attended only to the first principles of the logarithmic Tables. It is plain that all equations of the third degree, or cubics, may be distributed into three general classes, represented as follows: —

First class, ... $x^3 + ax = b.$

Second class, . . . $x^3 + a'x^2 = b'.$

Third class, ... $x^3 + ax^2 + a_1 x = b.$

To change an equation of the second class into the form of the first, let $x = \frac{1}{y}$. Substituting this value for x, and reducing,

$$y^3 - \frac{a'}{b'} y = \frac{1}{b'}.$$

The value of y can here be found, as in the first class, the reciprocal of which will be the value of x.

Again, by adding and subtracting equal quantities, the equation of the third class can take the form,

$$\left(x + \tfrac{a}{3}\right)^3 + \left(a_1 - \tfrac{a^2}{3}\right)\left(x + \tfrac{a}{3}\right) = b + \tfrac{a\,a_1}{3} - \tfrac{2\,a^3}{27}.$$

Making $x + \frac{a}{3} = z$, we may find the value of z, as in the first class, and then determine x from the relation

$$x = z - \tfrac{a}{3}.$$

Referring now to the investigation in the Mathematical Monthly, Vol. II., p. 85 – 88, let l denote the common logarithm of the quantity before which it is placed; and the following rule is obtained.

For the first class of cubics having the form,

$$x^3 + a\,x = b,$$

where a, b denote either positive or negative numbers.

Regarding a, b as positive numbers, find the quantity

$$L = 1\tfrac{1}{2} \,.\, l\,\tfrac{a}{3} + 10 - l\,\tfrac{b}{2}.$$

I. *When* a *in the given equation is positive;* find

$$l \tan v = L, \qquad l \tan u = \tfrac{1}{3}\left(20 + l \tan \tfrac{v}{2}\right),$$

$$l\,x = l\,2 + \tfrac{1}{2}\,l\,\tfrac{a}{3} + l \cot 2\,u - 10.$$

II. *When* a *is negative,* and the index of L less than 10;

$$l \sin v = L, \qquad l \tan u = \tfrac{1}{3}\left(20 + l \tan \tfrac{v}{2}\right),$$

$$l x = l 2 + \tfrac{1}{2} . l \tfrac{a}{3} + 10 - l \sin 2 u.$$

III. *When* a *is negative,* and the index of L is 10, or more;

$$l \cos v = 20 - L, \qquad l x = l 2 + \tfrac{1}{2} . l \tfrac{a}{3} + l \cos \tfrac{v}{3} - 10.$$

Here it should be carefully noted to *apply to the values of* x, *thus far, the same algebraic sign with that of* b, *on the right-hand side of the given cubic.* But the two remaining values of x should be taken with a sign contrary to that of b:

$$l x = l 2 + \tfrac{1}{2} l \tfrac{a}{3} + l \cos \left(60° - \tfrac{v}{3}\right) - 10,$$

and

$$l x = l 2 + \tfrac{1}{2} l \tfrac{a}{3} + l \cos \left(60° + \tfrac{v}{3}\right) - 10.$$

With the foregoing precepts, the auxiliaries v, u must always be taken positively, and not exceeding 90°.

Example.

Find the value of x, when $x^3 + 15 x = 4$.

[To find L and u.]		[To find x.]	
$l\left(\frac{a}{3} = 5\right) =$	0.698970	$l 2 =$	0.301030
$\frac{1}{2} l \frac{a}{3} =$	0.349485		0.349485
	+10.	$l \cot 2 u =$	8.773405
$- l\left(\frac{b}{2} = 2\right) =$	−(0.301030)		−10.
$l \tan v$, 79° 51′ 28″....	10.747425 $= L$.	l. 265412	1.423920
$l \tan \frac{v}{3}$, 39° 55′ 44″, 3	9.922720		
	20.	$x = +0.265412$.	
$l \tan u$, 43° 18′ 6″.5....	9.974240		
$2 u = 86° 36′ 13″$.			

In this example, a or 15 is positive, showing that the constant L is to be considered a logarithmic tangent, and that the formulas of I.

are to be employed; this has been strictly done in the above operation, as will be seen by comparison. And since b or 4 is positive, the value of x in this case is also positive.

Further examples might easily be multiplied, were it necessary; but the preceding rule, within a short compass, appears sufficiently plain and comprehensive. — L. W. Meech, Assistant, U. S. Coast Survey, Preston, Ct.

ON THE INDETERMINATE ANALYSIS.

By Rev. A. D. Wheeler, Brunswick, Maine.

[Concluded from Page 195.]

Proposition XV. The whole number of possible equations in such a series, or for every integral value of $c < ab + 1$ is expressed by the formula $\frac{ab-(a+b-1)}{2}$.

Demonstration. Dividing ab by a, we have $b =$ the whole number of impossible equations in which c is a multiple of a; and dividing by b, we have $a =$ the whole number of impossible equations in which it is a multiple of b. (Prop. IX., Cases 3 and 4.) But it will be seen that the impossible equation $ax + by = ab$ has thus been reckoned twice, when it should have been reckoned but once. Deducting 1 on account of this repetition, we shall have $a + b - 1$ for the whole number of impossible equations in which c is a multiple of either a or b. If this number be subtracted from ab, there will remain $ab - (a + b - 1)$, which, according to Prop. XIV., includes an equal number of possible and of impossible equations. Therefore $\frac{ab-(a+b-1)}{2}$ expresses all the possible equations in which $c < ab + 1$.

Prop. XVI. The whole number of equations in such a series, which will admit of one, and only one solution, is denoted by the product ab.

Dem. All values of $c > 2ab$ will admit of two solutions, according to Prop. VIII. It may be shown, as in the preceding proposition, that all the values of $c < 2ab + 1$, which will admit of two solutions, may be expressed by the same formula, $\frac{ab-(a+b-1)}{2}$; and this, subtracted from ab, representing the whole interval between ab and $2ab$, gives $ab - \frac{ab-(a+b-1)}{2} = \frac{ab+a+b-1}{2}$ for the whole number of equations within that interval, admitting of but one solution. Adding the expressions for the number of solutions admitting one solution only above ab and below it, we have

$$\frac{ab+(a+b-1)}{2} + \frac{ab-(a+b-1)}{2} = ab;$$

which was to be shown.

Prop. XVII. Let there be any numbers of independent equations, having the same coefficients to x and y. The sum of all the solutions contained in the whole number of equations may be found by dividing the sum of the values of c by ab, and adding the number of solutions contained in the several remainders, according to the rule which is given under Prop. XIII.

Adding the equations

$$ax + by = nab + r, \quad (A)$$
$$ax' + by' = n'ab + r', \quad (B)$$
$$ax'' + by'' = (n'' - 1)ab + r'' \text{ &c.} \quad (C)$$

we obtain

$$ax_1 + by_1 = ab(n + n' + n'' - 1) + r + r' + r'' \text{ &c.} \quad (D)$$

Subtracting the remainders, we have the equation,

$$ax_{,,} + by_{,,} = ab(n + n' + n'' - 1) = c;$$

and dividing c by ab, we obtain the quotient $n + n' + n'' - 1$, which gives the number of solutions in (D) with reference to the

remainders. But in equation (A) we have n solutions; in equation (B), n' solutions; and in equation (C), $n'' - 1$ solutions; and the sum of these $= n + n' + n'' - 1$, the same expression which we obtained from (D). Now if we ascertain the number of solutions in these several remainders, and add it to the number obtained from (D), we shall have all the possible solutions belonging to the several equations (A), (B), and (C).

Remark.— Several of the preceding propositions will be found very convenient in proposing questions which shall have any desired conditions, and several of them will very much facilitate the operations which are necessary to be performed in the solution of problems which are to follow. They may easily be verified by substituting numbers for letters. YOUNG'S Algebra contains more than one erroneous proposition relating to this subject, to which demonstrations are professedly given both by the author and the American editors.

PROBLEMS.

I. To find the successive remainders in any Arithmetical Progression, without continuing the operation of division.

1. Suppose the progression to be ascending, and call the first remainder r.

The general expression for any term in this progression is $a + (n-1)\,d$; and that for the next succeeding term will be $a + (n-1)\,d + d$. Let D be the divisor, and let

$$\frac{a + (n-1)\,d}{D} = Q + \frac{r}{D}.$$

Then will $a + (n-1)\,d = DQ + r$. By substitution, we have

$$\frac{a + (n-1)\,d + d}{D} = \frac{DQ + r + d}{D} = Q + \frac{r + d}{D}.$$

That is to find the next succeeding remainder, add the difference of

the progression to the remainder already found, rejecting the divisor as often as necessary.

2. Suppose the progression to be descending. Then by a process entirely similar, the remainder of the next succeeding term will be $r - d$. That is, from the remainder already found, the difference of the progression must be subtracted, and the divisor added as often as necessary. The reason for rejecting or adding the divisor may be readily shown. For $\frac{r+d-D}{D}$ still gives the same remainder $r+d$, and $\frac{D+r-d}{D}$ the same remainder $r - d$.

Illustration. — Let $D = 11$, $d = 7$, $a = 16$, $n = 8$. Then we have the progression

$$16,\quad 23,\quad 30,\quad 37,\quad 44,\quad 51,\quad 58,\quad 65\,;$$

and the remainders will be

$$5,\quad 1,\quad 8,\quad 4,\quad 0,\quad 7,\quad 3,\quad 10.$$

These are readily obtained by the rule, as follows: $16 - 11 = 5$; $5 + 7 - 11 = 1$; $1 + 7 = 8$; $8 + 7 - 11 = 4$; $4 + 7 - 11 = 0$; $0 + 7 = 7$; $7 + 7 - 11 = 3$; $3 + 7 = 10$.

Reversing the progression, we obtain the same remainders by reversing the process.

This method will serve to facilitate the operations, especially when the terms of the progression are large.

If we wish to find any particular term of these remainders, without finding all that precede, we have merely to make use of the formula

$$\frac{a+(n-1)\,d}{D} = Q + \frac{r}{D},$$

and r will be found at once. For example,

Let $n = 6$.

Then we have $\frac{5+5.7}{11} = 3 + \frac{7}{11}$.

Therefore we have for the 6th term, $r = 7$, which is the same as that given above.

II. To solve the equation $ax \pm by = c$.

FIRST METHOD.

When a or b is a divisor of c, and one is prime to the other.

As a will divide c, it must divide its equal, $ax + by$; and as it divides ax, it must also divide by (Prop. I.). But it cannot divide b, since a and b are prime to each other. Therefore it must divide y (Prop. III.), and the least value of y will be a; the next, $a + a$, or $2a$; the next, $3a$, &c.

Hence one of the values of x or y may be determined by simple inspection, and the corresponding value of the other unknown may be readily deduced from the equation. Then by the addition or subtraction of a or b, as the case requires, all the other values may be obtained.

When a or b is not a divisor of c, but has a factor in common with it, then it may be shown in the same way that y is equal to that factor or to some multiple of it.

In the first case, however, we must have $c > ab$ (Prop. IX., Case 3), or the solution in whole numbers will not be possible.

Examples.

Let $5x + 7y = 40$, then $y = 5$ and $x = 1$.

Let $6x + 7y = 39$, then $y = 3$, which is a factor of 6 and of 39.

SECOND METHOD.

When a, b, and c are all prime to each other, and the numbers are not large.

Let $ax + by = c$. Then $x = \frac{c - by}{a} =$ an integer. Divide c

by a and note the remainder. Find the successive remainders resulting from the division of by by a, as in Problem I. The term which gives the same remainder as c is the value of y.

As $\frac{c-by}{a}$ is the same with $-\frac{by-c}{a}$, the signs of this expression may be changed when more convenient.

Again, let $ax - by = c$. Then will $x = \frac{c+by}{a}$; and the value of y may be found in the same manner, except that it will be necessary to find a remainder $= a - r$, in order to cancel the remainder resulting from c.

Examples.

1. Let $11x + 13y = 141$. Then

$$x = \frac{141-13y}{11} = 12 - y + \frac{9-2y}{11}.$$

Substituting for y the successive values 1, 2, 3, &c., we find that $y = 10$ will give the remainder 9. Hence 10 is its true value. The remainders, as found by Problem I., are 2, 4, 6, 8, 10, 1, 3, 5, 7, 9; and 9 is found in the 10th term of the series.

2. Let $17x - 23y = 7$. Then $x = \frac{7+23y}{17}$, and we require the remainder $17 - 7 = 10$. The series will be 6, 12, 17, 13, 2, 8, 14, 3, 9, 15, 4, 10. The remainder 10 is found in the 13th term. Therefore $y = 13$. This method is expeditious when the divisor is small.

THIRD METHOD.

1. To solve the indeterminate equation $ax - by = 1$.

By the theory of Continued Fractions, if any two consecutive approximations be reduced to a common denominator, the difference of their numerators will always be 1; and this difference will be positive, or negative, according as the number of quotients in the reduction is even or odd.

If a and b are prime to each other, and the fraction $\frac{a}{b}$ be reduced

to the form of a continued fraction, and the several approximate values obtained, the least value will be the fraction $\frac{a}{b}$ itself; and the next preceding may be represented by the ascertained values $\frac{y'}{x'}$. Now, in accordance with what has been stated,

$$\frac{a}{b} - \frac{y'}{x'} = \frac{ax' - by'}{bx'} = \frac{\pm 1}{bx'};$$

or, rejecting the denominators, $ax' - by' = \pm 1$, the upper sign denoting that the number of quotients in obtaining x' and y' is even, the lower that it is odd.

If then we make $x = x'$ and $y = y'$, we have in the first case $ax' - by' = 1$; and the problem is solved. In the second case, we have $ax' - by' = -1$. But by changing x' into $b - x'$, and y' into $a - y'$, we obtain

$$ax - by = a(b - x') - b(a - y') = -ax' + by' = +1;$$

and this case is also solved.

Hence the following rule: —

Find x' and y' by the process for finding the greatest common divisor, and the rule for Continued Fractions. Then, if the number of quotients be even, x' will be the value of x, and y' will be that of y; but if the number be odd, $b - x'$ and $a - y'$ will be their respective values.

Examples.

1. $12x - 17y = 1$.

12	17	1	
10	12	2	
2	5	2	
	4		$5 = y'$
	1		$7 = x'$

Here the number of quotients is odd. Therefore $x = 17 - 7 = 10$, and $y = 12 - 5 = 7$. Whence we have

$$ax - by = 12 \,.\, 10 - 17 \,.\, 7 = 1,$$

$$2 \,.\, 2 + 1 = 5 = y', \quad \text{and} \quad 5 \,.\, 1 + 2 = 7 = x'.$$

2. $10x - 13y = 1$.

10	13	1	
9	10	3	$= y'$
1	3	4	$= x'$

Here the number of quotients is even. Hence, $x = x' = 3 \cdot 1 + 1 = 4$, and $y = y' = 3$.

Rule. — Multiply the last quotient by that which immediately precedes, and add 1. Multiply this result by the last quotient not used, and add the preceding sum. Proceed in this manner till all the quotients have been employed. The last result will be x'; the one before it y', When there are but two quotients, there will be but one multiplier, and consequently the latter of the two will express the value of y'.

The whole operation is simple when once understood, and may be performed mentally, if the numbers are not too large.

2. To solve the equation $ax - by = c$.

Find x', and y', as before. Then will $ax' - by' = 1$, or $-ax + by = 1$, according as the number of quotients is even or odd.

1. Let $x = cx'$, and $y = cy'$. The equation becomes

$$ax - by = acx' - bcy' = c(ax' - by') = c;$$

and the conditions of it are satisfied.

2. Let $x = c(b - x')$, and $y = c(a - y')$. Then we have

$$ax - by = ca(b - x') - cb(a - y') = c(-ax' + by') = c;$$

and the conditions in this case are also satisfied.

3. To solve the equation $ax + by = nab + r$.

Find x', and y', as before; and suppose the number of quotients to be even. Let $x = rx'$, and $y = na - ry$. Then will

$$ax + by = arx' + nab - bry' = nab + r(ax' - by') = nab + r;$$

and this case is solved.

Suppose the number of quotients to be odd, and let $x = nb - rx'$, and $y = ry'$. Then

$$ax + by = nab - arx' + bry = nab + r(-ax' + by') = nab + r.$$

Wherefore this case also is solved.

The following formulæ are sufficient for the solution of all these cases: —

1. $ax + by = nab + r$; $x = rx'$ and $y = na - ry'$; when the quotients are *even*.
2. $ax + by = nab + r$; $x = nb - rx'$ and $y = ry'$; when the quotients are *odd*.
3. $ax - by = c$; $x = cx'$ and $y = cy'$; when the quotients are *even*.
4. $ax - by = c$; $x = c(b - x')$ and $y = c(a - y')$; when the quotients are *odd*.

Examples.

1. $8x + 13y = 319 = \overset{n}{3} \,.\, \overset{a}{8} \,.\, \overset{b}{13} - +\overset{r}{7}$

1	
1	$x = rx' = 35.$
1	
1	$y = na - ry' = 24 - 21 = 3.$
2	
$3 = y'$	$8x + 13y = 8 \,.\, 35 + 13 \,.\, 3 = 319.$
$5 = x'$	

2. $10x + 17y = 689 = \overset{n}{4} \,.\, \overset{a}{10} \,.\, \overset{b}{17} + \overset{r}{9}.$

1	
1	$x = nb - rx' = 68 - 45 = 23.$ $\quad y = ry' = 27.$
2	
$3 = y'$	$ax + by = 10 \,.\, 23 + 17 \,.\, 27 = 689.$
$5 = x'$	

3. $\overset{a}{7}x - \overset{b}{10}y = \overset{c}{11}.$

1	$x = cx' = 33.$ $\quad y = cy' = 22.$
$2 = y'$	
$3 = x'$	$ax - by = 7 \,.\, 33 - 10 \,.\, 22 = 11.$

4. $7x - 12y = 5.$

1	$x = c(b - x') = 5(12 - 5) = 35.$
1	
2	$y = c(a - y') = 5(7 - 3) = 20.$
$3 = y'$	
$5 = x'$	$7x - 12y = 7 \,.\, 35 - 12 \,.\, 20 = 5.$

4. Problem relating to three unknown quantities.

To determine the number of solutions which the equation

$$ax + by + cz = d,$$

will admit of in integers.

SOLUTION.

If we put for z the numbers 1, 2, 3, &c., in their natural order, we shall obtain the equations,

$$ax + by = d - c,$$
$$ax' + by' = d - 2c,$$
$$ax'' + by'' = d - 3c, \text{ \&c.}$$

until we obtain, at length,

$$ax_{,} + by_{,} = d - nc < a + b,$$

which will not admit of a solution.

Now the number of solutions which each of these equations can have, may be readily ascertained by the rule under Prop. XVI.; and as we give to z all its possible values, we shall evidently, by adding together the number belonging to each successive equation, obtain all the solutions required by the problem.

Example.

Let $3x + 5y + 7z = 100$. Giving to z the values of 1, 2, 3, &c., in succession, we have the equations,

$$3x + 5y = 93,$$
$$3x' + 5y' = 86,$$
$$3x'' + 5y'' = 79, \text{ \&c.}$$

Now,

$\frac{d-c}{ab} = 93 \div 15 = 6$, with the remainder 3, and gives 6 solutions.

$\frac{d-2c}{ab} = 86 \div 15 = 5$, " " 11, " 6 "

$\frac{d-3c}{ab} = 79 \div 15 = 5$, " " 4, " 5 "

&c. $72 \div 15 = 4$, " " 12, " 4 "

$65 \div 15 = 4$, " " 5, " 4 "

$58 \div 15 = 3$, " " 13, " 4 "

$51 \div 15 = 3$, " " 6, " 3 "

$44 \div 15 = 2$, " " 14, " 3 "

$37 \div 15 = 2$, " " 7, " 2 "

$30 \div 15 = 2$, " " 0, " 1 "

$23 \div 15 = 1$, " " 8, " 2 "

$16 \div 15 = 1$, " " 1, " 1 "

$9 \div 15 = 0$, " " 9, " 0 "

Wherefore the whole number of solutions is . . . 41

When d is very large, and a, b, and c are very small, this method becomes laborious. It may, however, be greatly simplified.

1. The terms $d - c$, $d - 2c$, $d - 3c$, &c. are those of Arithmetical Progression; and as we have the first term, last term, and difference, we can easily find the sum of all the terms. Call this sum S.

2. As the remainders must necessarily recur in periods (Prop. V., R. 2), and as it is easy to find all the remainders in a period (Problem I.), we can readily obtain the sum of all the remainders, by multiplying the sum in one period by the number of periods and adding those in the partial period. Call this sum S'.

3. The number of solutions contained in the remainders may be found in the same way, by first finding those in one period, and then multiplying by the number of periods. Call it S''.

Then the whole number of solutions $= \frac{S - S'}{ab} + S''$.

When the period returns after ab terms, as it often does, the sum

of the remainders in a period will be equal to the sum of an arithmetical progression, whose first term is one, and difference one, and number of terms ab. It will therefore be expressed by the formula $\frac{ab\,(ab+1)}{2}$. Also the number of possible solutions in a period will be expressed by the formula $\frac{ab-(a+b-1)}{2}$ (Prop. XV.).

Thus, taking the last example,

$3x+5y+7z=100$

$3x+5y=\ 93=l$

$2=a$

$95=a+l$

$7=\frac{1}{2}n$

665

110

15)555

37

4

41

$7)93$

$13-2$

$n=14,\quad a=2$

$\frac{15-(3+5-1)}{2}=4$

$15=ab$

1

$2)16=ab+1$

$8=\frac{ab+1}{2}$

15

120 $\quad\frac{ab\,(ab+1)}{2}$

10

110

Example 2. To find the whole number of solutions that the equation $7x+9y+23z=999$ will admit of.

Operation. — Calling $z=1$, we have

$7x+9y=9976=l, \qquad 9976\div 23=433\frac{17}{23}, \qquad \therefore\ n=434;$

since the remainder 17 also constitutes a term, $\therefore\ a=17$. But $(a+l)\frac{n}{2}=S=(17+9976)\,.\,217=2168481, \quad 434\div 63=6\frac{56}{63}$; that is, 6 complete periods of remainders, and 56 remainders belonging to another period, or $63-56=7$ remainders that are wanting to complete another period.

$(63+1)\div 2=32.$

$32\,.\,63=2016=$ the sum of the remainders in one period.

$2016\,.\,6=12096=$ the sum in six periods.

$(9976 - 62 \cdot 23) \div 63 = 135\frac{45}{63}$.

$\therefore$ 45 = the last remainder in a period, as 22 = the first.

Then, by Problem I., the 7 previous remainders will be

$$45 + 5 + 28 + 51 + 11 + 34 + 57 = 231.$$

Then $2016 - 231 = 1785$ = the sum of the 56 remainders in the partial period, and $12096 + 1785 = 13881 = S'$ = the sum of all the remainders. Then

$S - S' = 2168481 - 13881 = 2154600$. $(S - S') \div ab = 2154600 \div 63 = 34200$ = the number of solutions, exclusive of those comprehended in the remainders.

$\frac{ab - (a + b - 1)}{2} = (63 - (7 + 9 - 1)) \div 2 = 24$ = the number of solutions in the remainders of one period.

$24 \cdot 6 = 144$ = the number in 6 periods.

$24 - 3$ (3 solutions are contained in the 7 remainders) $= 21$ = the number of solutions in the partial period.

Then

$144 + 21 = 165 = S''$; and $34200 + 165 = 34365$ = the answer.

5. To find the least whole number, which, being divided by given numbers, shall leave given remainders.

Suppose the number to be x, the divisors d, d', d'', &c., and the remainders r, r', r'', &c. Then each of the quantities, $\frac{x - r}{d}$, $\frac{x - r'}{d'}$, $\frac{x - r''}{d''}$, &c., is an integer. Reduce them to a common denominator, and they become

$$\frac{d' d'' x - d' d'' r}{d d' d''}, \qquad \frac{d d'' x - d d'' r}{d d' d''}, \qquad \frac{d d' x - d d' r}{d d' d''};$$

and having the same values, they still remain integers. Now, whether we add or subtract these several quantities, or multiply them by any whole number or divide them by any exact divisor, the

sums, differences, products, and quotients will also be integers. (See the Axioms.)

Suppose, then, we subtract the third from the second, and this remainder from the first. The result will be,

$$\frac{d' d'' x - d d'' x + d d' x - d' d'' r + d d'' r - d d' r}{d d' d''}, \text{ an integer.}$$

This will fulfil all the conditions. For, dividing by d, we obtain

$$-d'' x + d' x + d'' r - d' r + \frac{d' d'' x - d' d'' r}{d} = \frac{d' d'' (x - r)}{d};$$

the possibility of which depends, inasmuch as d, d', d'' are supposed to be prime to each other, upon the original supposition that $\frac{x - r}{d}$ is an integer.

In the same manner, it may be shown that all the other conditions will be fulfilled.

Rule. — Reduce the fractions $\frac{x - r}{d}$, $\frac{x - r'}{d'}$, $\frac{x - r''}{d''}$, &c., to equivalent fractions having a common denominator. Set the numerators in a column, one under another, the largest at the top, as follows: —

A	$d' d'' x - d' d'' r,$
B	$d d'' x - d d'' r',$
C	$d d' x - d d' r'',$ &c.

Subtract C from B, and this remainder from A. Thus bring all the numerators in the operation. Then continue subtracting these remainders, or some multiples or exact quotients of them, one from another, until x is obtained with unity for a coefficient; and rejecting the least common denominator from the number on the right as often as there is an excess. Then the last value on the right, if negative, is the answer sought. Otherwise it must be subtracted from the denominator, and the remainder will be the answer.

Example.

What number is that which, being divided by 11, 19, and 29, will leave the remainders 3, 5, and 10?

$\frac{x-3}{11}$, $\frac{x-5}{19}$, $\frac{x-10}{29}$, are integers,

or,

$\frac{551x-1653}{6061}$, $\frac{319x-1595}{6061}$, $\frac{209x-2090}{6061}$

A,	$551x - 1653$
B,	$319x - 1595$
C,	$209x - 2090$
B — C,	$110x + 495$
A — (B — C),	$441x - 2148$
4 (B — C),	$440x + 1980$
	$x - 4128$

4128 = the answer.

The reason for taking 4128 as the value of x, as also for subtracting the number, when positive, from the common denominator, may be found by referring to the second method under the second Problem: the full expression in this case being $\frac{x-4128}{6061}$.

Problems of this kind may also be solved after reducing them to a common denominator, and combining the numerators in any manner thought best, according to the third method under Problem II.

Take, for illustration, from the previous example the quantity expressed by A — (B — C), $\frac{441x-2148}{6061} = y$. Then

$$\overset{a}{441}x - \overset{b}{6061}y = \overset{c}{2148}.$$

3
1
2
1
9
3
1
4
37
41
119
160 = y'
2199 = x'

Then, by formula 4, as the number of quotients is odd, we have

$$x = c(b - x') = 2148(6061 - 2199) = 8295576.$$

This is not the least value of x, but that value + some multiple of b. (See Prop. VIII.) We must therefore reject this mutiple of b; or, in other words, divide 8295576 by 6061, and there will remain 4128, as before.

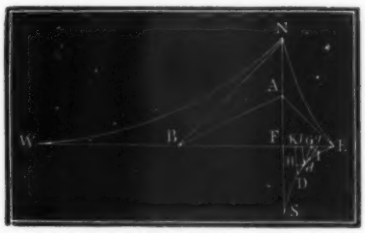
For this example, however, the former method is more concise.

SOLUTION OF A PROBLEM IN "CURVES OF PURSUIT."

By Dr. Joel E. Hendricks, Newville, Indiana.

Problem. — Three foxes are supposed to be at F, and a dog at S, forty rods south of F. They all start at the same instant, the foxes running with the same uniform velocity, and the dog just twice as fast. The first fox runs east, along the line FE; the second one runs north, along the line FA; and third runs west, along the line FW. The dog runs directly towards the first fox, and overtakes it at E. He immediately pursues the second one, and overtakes it at N; then pursues the third, and overtakes it at W. How far must the dog run to catch the three foxes? — Communicated by Prof. Root (Vol. I., p. 249).

Solution. — Let D represent the position of the dog when the first fox is at G, and draw DK parallel to SF, and dk infinitely near and parallel to DK. Also draw DG tangent to the curve at D, and dg tangent at the point d, and draw GI perpendicular to dg, and dH perpendicular to DK. Put $SF(=40)=a$, $FK=w$, $FG=x$, $DG=y$, and curve $SD=z$; and let the corresponding lines for the second and third curves be denoted by w_1, x_1, y_1, z_1, and w_2, x_2, y_2, z_2. Then is $Kk=dw$, $Gg=dx$, $Ig-Dd=dy$, and $Dd=dz$; and by the similar triangles HDd and IGg we have



$$dz : dw :: dx(=\tfrac{1}{2}dz) : \tfrac{1}{2}dw = Ig. \quad \therefore dy = \tfrac{1}{2}dw - dz.$$

But $dz = 2dx$, $\therefore dy = \tfrac{1}{2}dw - 2dx$, $\therefore dx = \tfrac{1}{4}dw - \tfrac{1}{2}dy$. Whence by integrating we have

$$(1) \qquad x = \tfrac{1}{4}w - \tfrac{1}{2}y + C.$$

Similarly,

(2) $x_1 = \frac{1}{4} w_1 - \frac{1}{2} y_1 + C_1$. (3) $x_2 = \frac{1}{4} w_2 - \frac{1}{2} y_2 + C_2$.

When $w = 0$ in (1),

$$x = 0, \qquad y = a; \qquad \therefore\ C = \tfrac{1}{2} a.$$

When $w_1 = 0$ in (2),

$$x_1 = FE, \qquad y_1 = FE\sqrt{2}; \qquad \therefore\ C_1 = \tfrac{1}{2} FE(2+\sqrt{2}).$$

When $w_2 = 0$ in (3),

$$x_2 = FN, \qquad y_2 = FN\sqrt{2}, \qquad \therefore\ C_2 = \tfrac{1}{2} FN(2+\sqrt{2})^2.$$

By substituting for C, C_1, C_2, in (1), (2), (3), we have

(4) $$x = \tfrac{1}{4} w - \tfrac{1}{2} y + \tfrac{1}{2} a.$$

(5) $$x_1 = \tfrac{1}{4} w_1 - \tfrac{1}{2} y_1 + \tfrac{1}{2} FE(2+\sqrt{2}).$$

(6) $$x_2 = \tfrac{1}{4} w_2 - \tfrac{1}{2} y_2 + \tfrac{1}{2} FN(2+\sqrt{2}).$$

Now, when the dog overtakes the fox in running on each of the curves, the points D and G will coincide; and consequently y will become zero, and w will become x. Hence, if x', x_1', x_2' denote the whole distance run by the three foxes respectively, we shall have,

$$x' = \tfrac{1}{4} x' + \tfrac{1}{2} a; \qquad \therefore\ x' = \tfrac{2}{3} a.$$

$$x_1' = \tfrac{1}{4} x_1' + \tfrac{1}{3} a(2+\sqrt{2}); \qquad \therefore\ x_1' = \tfrac{4}{9} a(2+\sqrt{2}).$$

$$x_2' = \tfrac{1}{4} x_2' + \tfrac{2}{9} a(2+\sqrt{2})^2; \qquad \therefore\ x_2' = \tfrac{8}{27} a(2+\sqrt{2})^2.$$

By the question,

$$z' = 2x' = \tfrac{4}{3} a = 53.333$$

$$z_1' = 2(x_1' - x') = 2a\left(\tfrac{4}{9}(2+\sqrt{2}) - \tfrac{2}{3}\right) = \tfrac{4}{9} a(1+\sqrt{2}) = 68.061$$

$$z_2' = 2(x_2' - x_1') = 2a\left(\tfrac{8}{27}(2+\sqrt{2})^2 - \tfrac{4}{9}(2+\sqrt{2})\right) = \tfrac{8}{27} a(6+5\sqrt{2}) = 154.916$$

Therefore the whole distance run by the dog = 276.310 rods.

Second Solution, by Prof. O. Root, Hamilton College, Clinton, N.Y.

Take the origin at F, and let xy be the co-ordinates of any point in the curve described by the dog in the pursuit; then will

$\int (dx^2 + dy^2)^{\frac{1}{2}}$ be the space passed over by the dog, and $x - \frac{y\,dx}{dy}$ that passed over by the first fox in the same time; but as these spaces are as 2 to 1, we have

$$x - \frac{y\,dx}{dy} = \tfrac{1}{2}\int (dx^2 + dy^2)^{\frac{1}{2}}.$$

By differentiating and reducing, the equation becomes

$$-\frac{d^2x}{(dx^2+dy^2)^{\frac{1}{2}}} = \tfrac{1}{2}\frac{dy}{y};$$

and by integrating,

$$(1) \qquad \log\left(\frac{dx}{dy} + \frac{(dx^2+dy^2)^{\frac{1}{2}}}{dy}\right) = -\tfrac{1}{2}\log y + C.$$

If now we put $SF = 40$ rods $= a$, then, when $y = a$, we have $dx = 0$, and hence $C = \frac{1}{2}\log a$; and therefore (1) becomes

$$\log\left(\frac{dx}{dy} + \frac{(dx^2+dy^2)^{\frac{1}{2}}}{dy}\right) = \log\frac{a^{\frac{1}{2}}}{y^{\frac{1}{2}}};$$

or,
$$\frac{dx}{dy} + \frac{(dx^2+dy^2)^{\frac{1}{2}}}{dy} = \frac{a^{\frac{1}{2}}}{y^{\frac{1}{2}}};$$

or,
$$2\,dx = -\frac{a^{\frac{1}{2}}}{y^{\frac{1}{2}}}\,dy + \frac{y^{\frac{1}{2}}}{a^{\frac{1}{2}}}\,dy;$$

or, (2)
$$2x = -2\,a^{\frac{1}{2}}\,y^{\frac{1}{2}} + \tfrac{2}{3}\frac{y^{\frac{3}{2}}}{a^{\frac{1}{2}}} + C'.$$

When $y = a$, $x = 0$; $\therefore\ C' = \frac{4a}{3}$, and (2) becomes

$$2x = 2\,a^{\frac{1}{2}}\,y^{\frac{1}{2}} + \tfrac{2}{3}\frac{y^{\frac{3}{2}}}{a^{\frac{1}{2}}} + \frac{4a}{3},$$

which is the equation of the curve. When the dog overtakes the fox $y = 0$, and therefore $2x = \frac{4a}{3} = 53\frac{1}{3}$ rods for the distance run to catch the first fox.*

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* We omit the remainder of the solution, as the student will find no difficulty in completing it. It will be observed that Dr. Hendrick's solution involves a differential equation of only the first order, while Prof. Root's, which is the solution usually given, involves one of the second order. Is Dr. Hendrick's ingenious solution new? We have never seen it before. — Editor.

NOTE ON DIACAUSTICS.

By Rev. Thomas Hill, President of Antioch College, Yellow Springs, Ohio.

While acknowledging the justice of Mr. Paterson's remarks (Math. Monthly, p. 40, Vol. I.) on the waste of time involved in rediscovering that which was known centuries ago, I beg permission also to amuse myself with mathematical trifles even when I have not time to look up the results of previous investigators. The following note on Caustics is offered to the "Monthly," because it is interesting to me; but I do not know whether any or all of it is to be found in other places, with the exception of the corollary concerning the boundary of the space run through by a line in a rolling circle, which may be found, having no connection with caustics, in Chasles.

1. Let $\rho = f\nu$ be the equation of any plane curve, ρ being the radius of curvature, and ν the angle it makes with a fixed axis. Let θ be the angle which parallel rays of light make with the axis of ν.

2. If the distance from the curve to the caustic, measured on the reflected ray, be called d, we have, by trigonometry,

$$\frac{d}{\rho\, d\nu} = \frac{\sin(\frac{1}{2}\pi - (\theta - \nu))}{\sin 2\, d\nu},$$

$$d = \tfrac{1}{2}\rho \cos(\theta - \nu). \tag{1}$$

3. The element of the caustic is

$$d\,d + \rho\, d\nu \sin(\theta - \nu) = \tfrac{1}{2} d\rho \cos(\theta - \nu) + \tfrac{1}{2}\rho \sin(\theta - \nu)\, d\nu;$$

and this divided by $2\, d\nu$ gives the radius of the caustic,

$$\rho_1 = \tfrac{1}{4}(D\rho \cos(\theta - \nu) + 3\rho \sin(\theta - \nu)). \tag{2}$$

4. From (1) it follows that a caustic is at its maximum distance from the generating curve, when the parallel rays strike the curve at right angles, the distance being half the radius of curvature of the curve.

5. From (2) it follows that the radius of a caustic, when parallel rays strike the generating curve at right angles, is one fourth the radius of the evolute of the curve. But where the rays strike the curve tangentially, the radius of the evolute is three fourths that of the curve itself.

6. Equation (2) may be reduced for any particular curve to the form $\rho_1 = f\nu_1$, by putting

$$\nu = \tfrac{1}{2}(\nu_1 + \theta - \tfrac{1}{2}\pi).$$

7. For example, let $\rho = 4R\sin\nu$; and we have

$$\begin{aligned}\rho_1 &= R\cos^2\nu\cos\theta + 4R\cos\nu\sin\nu\sin\theta - 3R\sin^2\nu\cos\theta \\ &= -R\cos\theta + 2R\cos\theta\sin(\nu_1+\theta) - 2R\sin\theta\cos(\nu_1+\theta) \\ &= 2R\sin\nu_1 - R\cos\theta.\end{aligned}$$

That is to say, the caustic of a cycloid, for parallel rays, is concentric with a cycloid of half the radius, coinciding with the small cycloid when $\theta = \frac{1}{2}\pi$.

8. Let the circle generating the original cycloid be again rolled through it, and it is manifest that the reflected ray will make, with a diameter through the generating point, the constant angle $\frac{1}{2}\pi - \theta$. Hence a diameter of the generating circle is a sliding tangent on the small cycloid, and any chord in the circle is a sliding tangent on a curve concentric with a half-size cycloid.

9. The caustic for $\rho = A\sin^n\nu$ is (see GOULD'S Astr. Journal),

$$\rho_1 = \tfrac{1}{4}A(n\sin^{n-1}\nu\cos(\theta-\nu)\cos\nu + 3A\sin^n\nu\sin(\theta-\nu));$$

which, by putting $\theta = \frac{1}{2}\pi$, reduces to

$$(3)\quad \rho_1 = \frac{3+n}{4}A\sin^n\nu\cos\nu = \left(\frac{1}{2}\right)^{\frac{n+1}{2}}\left(\frac{3+n}{4}\right)A(1-\cos\nu_1)^{\frac{n}{2}}(1+\cos\nu_1)^{\frac{1}{2}}.$$

10. The substitution in the last equation of $n = 1$ gives for the cycloid, as in Art. 7,

$$\rho_1 = \frac{A}{2}\sin\nu_1.$$

The substitution of $n = -3$ gives for the parabola a single point, $\rho_1 = 0$.

11. Equation (3) may also be written,

$$\rho_1 = \frac{3+n}{4} A \sin^n \tfrac{1}{2}\nu_1 \cos \tfrac{1}{2}\nu_1. \tag{4}$$

12. The integral of (4) gives, for the involute of the caustic,

$$\rho_2 = \frac{3+n}{2(n+1)} A \sin^{n+1} \tfrac{1}{2}\nu_1 = A_1 \sin^m \tfrac{1}{2}\nu_1.$$

13. The last equation shows that the involutes of the caustics of $\rho = A \sin^n \nu$ have the same form as the original curves, except that $\frac{1}{2}\nu$ is substituted for ν, and 1 is added to n. The caustics bear the same relation to the evolutes of the curves.

14. The substitution of $\frac{1}{2}\nu$ for ν in the equation of a curve will make it necessary for ρ to revolve twice as far before going through the same changes. This may be conceived as effected by either of the following metamorphoses.

Conceive the curve as composed of infinitesimal links, in which $d\nu$ is of equal value. Imagine these links capable of extension or retraction without altering their curvature, and it will be evident that by making them each a times as long as before, ν will change a times as much for a given change of ρ, so that the equation will be changed from $\rho = f\nu$ to $\rho = f\frac{\nu}{a}$. The extension may be conceived as effected by sliding joints.

Or imagine these links capable of being bent to a greater or less curvature, their length being unchanged; and it will be evident that by giving each link b times as much curvature, ν must change b times as much in order to produce the same order of changes in ρ, and at the same time ρ will be diminished to $\frac{\rho}{b}$. That is to say, $\rho = f\nu$ will become $\rho = \frac{1}{b} f \frac{\nu}{b}$.

Thus the first change will convert the cycloid $\rho = 4R \sin \nu$ into the epicycloid $\rho = 4R \sin \frac{\nu}{a}$, while the second change would convert the same cycloid into the epicycloid $\rho = \frac{4a}{b} \sin \frac{\nu}{b}$. Hence the

cycloid, by sliding into itself, or by extending itself, or by having the curvature of each part increased or diminished in proportion to the existing curvature, may be changed into any of the epicycloids and hypocycloids $\rho = A \sin a\nu$, in which $\frac{A\,a}{1-a^2}$ is the radius of the stationary, and $\frac{A}{2\,(1+a)}$ that of the generating.

15. Other examples of the peculiar deformation to which the caustics of $\rho = A \sin^n \nu$ have led us, are reserved for another note.

ON THE LOCUS OF PERPENDICULAR TANGENTS TO ANY CONIC SECTION.

By William Chauvenet, Professor of Mathematics in Washington University, St. Louis.

"Pairs of tangents perpendicular to each other, are drawn to any conic section; find the locus of their intersection."

The solution of this problem, where the given conic section is the ellipse, is well known. It was proposed and solved in a very elegant manner by Bessel, in the introduction to the volume of the Königsberg Observations for 1814, where it is applied to determining the effect of an elliptical form of the pivot of a transit instrument revolving in a V, the faces of which are perpendicular to each other. To obtain a general solution, I take the equation

$$y^2 = 2px + x^2,$$

which represents any conic section; the axis of x being the axis of the curve, the origin being the extremity of the axis of the curve (or vertex), $2p$ being the parameter, and n depending on the nature of the curve. If now a', a'' denote the trigonometric tangents of the angles which two tangent lines to the given curve make with the axis of x, (x', y') and (x'', y'') the points of contact, we have, for the equations of these tangent lines,

$$(1) \qquad -a'(x - x') + (y - y') = 0,$$

$$(2) \qquad -a''(x - x'') + (y - y'') = 0,$$

with the conditions,

$$a'a'' + 1 = 0, \tag{3}$$

$$a'y' = p + nx', \tag{4}$$

$$a''y'' = p + nx'', \tag{5}$$

$$y'^2 = 2px' + nx'^2, \tag{6}$$

$$y''^2 = 2px'' + nx''^2; \tag{7}$$

of which (4) and (5) are obtained by differentiating our general equation and putting a' and a'' for $D_{x'}(y')$ and $D_{x''}(y'')$.

From these seven equations we have to eliminate the six quantities $x', y', x'', y'', a', a''$, to obtain the equation of the required locus involving only x and y (which, being common to the two tangents, will represent their intersection), and the constants n and p.

Substituting in (1), (2), (6), and (7), the values of x' and x'', given by (4) and (5), we find

$$(a'^2 - n)\,y' = a'nx - ny + a'p, \tag{8}$$

$$(a''^2 - n)\,y'' = a''nx - ny + a''p, \tag{9}$$

$$(a'^2 - n)\,y'^2 = p^2, \tag{10}$$

$$(a''^2 - n)\,y''^2 = p^2; \tag{11}$$

from which we immediately deduce

$$(a'nx - ny + a'p)^2 - (a'^2 - n)\,p^2 = 0, \tag{12}$$

$$(a''nx - ny + a''p)^2 - (a''^2 - n)\,p^2 = 0, \tag{13}$$

or,

$$n(a'x - y)^2 + 2a'p(a'x - y) + p^2 = 0, \tag{14}$$

$$n(a''x - y)^2 + 2a''p(a''x - y) + p^2 = 0. \tag{15}$$

Multiplying (15) by a'^2, and observing the condition (3), we have

$$n(x + a'y)^2 + 2p(x + a'y) + a'^2p^2 = 0;$$

which, added to (14), gives

$$(nx^2 + ny^2 + 2px + p^2)(a'^2 + 1) = 0, \tag{16}$$

in which the factor a^2+1 cannot be zero; and hence we must have

(A) $$nx^2+ny^2+2px+p^2=0,$$

which is the equation of the required locus. If put under the form

(B) $$\left(x+\frac{p}{n}\right)^2+y^2=\frac{p^2}{n^2}(1-n)=R^2,$$

it shows that the locus is in general the circumference of a circle, the centre of which is in the axis at the distance $-\frac{p}{n}$ from the origin, and whose radius is $R=\frac{p}{n}\sqrt{1+n}$.

For the ellipse, the semi-axes of which are A and B, we have

$$n=-\frac{B^2}{A^2}, \qquad -\frac{p}{n}=A, \qquad R=\sqrt{A^2+B^2};$$

so that the centre of the circle is the centre of the ellipse, and the diameter of the circle is the diagonal of the rectangle continued by the axes of the ellipse.

For the hyperbola, the semi-axes of which are A and B, we have

$$n=+\frac{B^2}{A^2}, \qquad -\frac{p}{n}=-A, \qquad R=\sqrt{A^2+B^2};$$

and the centre of the circle is still the centre of the curve. But if $A<B$ there is no locus, since R is then imaginary. If $A=B$, $R=0$, and the locus becomes a point which is the centre of the hyperbola. That is, in the equilateral hyperbola only one pair of tangents perpendicular to each other can be drawn, and these are evidently the asymptotes.

For the parabola, we have

$$n=0, \qquad -\frac{p}{n}=\infty, \qquad R=\infty;$$

so that the locus is a straight line whose equation is, by (A),

$$x=-\frac{p}{2},$$

which is the equation of the directrix of the parabola.

This solution may interest the student as an example of elimination, the "method of substitution" leading directly to the required result.

THE CENTRAL SCHOOL OF ARTS AND MANUFACTURES AT PARIS.

[It gives us great pleasure to call attention to the following communication from Mr. Watson. His notes on the Polytechnic and other Schools of Paris, will be given in early numbers of Vol. III.]

History, General Plan, Conditions for Admission, Courses of Lectures, Drawings, Manipulations in the Laboratories of Chemistry and Physics, Discipline, etc.

Peclet, Dumas, and Olivier, founded the Central School in 1829. Peclet, a graduate of the Normal School, had been for some time a Professor of Physics at Marseilles; after the restoration, on account of his political opinions, he had lost his place, yet he still devoted himself to the industrial applications of his science.

Dumas, at that time a Repetiteur of Chemistry in the Polytechnic School, was already known by important scientific works. Olivier, also a graduate of the Polytechnic School, had just returned from Sweden, where he had reorganized the schools of Artillery, and founded a chair of Descriptive Geometry, applied to military service. These three men, theorists by their studies, had become practical from the nature of their works. They selected the name of Central School of Arts and Manufactures for two reasons, first because it had been the former name of the Polytechnic School, which they wished simply to re-establish according to the original design of the founders, and secondly that no one should be deceived as to their own intention. They framed for it the following plan, which is rigidly adhered to in every particular at present.

The basis of the teaching must be the study of Geometry, Mechanics, Physics, and Chemistry (since every engineer should possess an extended knowledge of these four sciences); and the teaching of them must always be conducted with a view to their application to industry. This will form the instruction for the first year, which is to be preparatory, and during which the student is fitted to enter upon the courses of the second and third years, which form, so to speak, the school of application. The professors must never forget that the object of the school is *to form instructed and capable engineers, and not savans.*

They will exhibit in detail, during the first year, those theories leading to the applications in industry, and for all the others, they will expose them summarily, so that the students may know what has been found out, and may know where they can study them in detail.

The professors of the first year must not be purely savans, but must be taken from the engineers, retired or in active service: in a word they should be, without ceasing, in contact with industry, and by profession or by taste should love or cultivate the applications of these sciences.

During the first year the students will execute drawings in Descriptive Geometry in all its applications, and also in Mechanics and Architecture.

They will make manipulations in Chemistry and Physics, the practice being always annexed to the oral exposition.

In the courses of the second and third years, the students will be shown the applications of the theories of the first year to different works of engineering.

The students will follow all the courses, but will only execute projects and manipulations in their speciality.

All the professors of the courses of the second and third years *must* be engineers, for it is only the man who has practised ("*Qui a mit la main a la pâte*") who is able to teach all the details of the construction of a bridge, of a water-wheel, of a railroad, of a steam-engine, in the best manner, and it also accomplishes the object of keeping the school up to the level of the requirements of industry. Young men are admitted between the ages of seventeen and thirty, from all countries, of all political opinions, and of every religion: by this means a desirable fusion is

established between the different political and religious parties which will eventually lead the way to industrial accommodations between the different nations.

The school in 1857 became an establishment of the state, but no modification has been made, unless it has been to raise a little the standard for graduation.

The following are the requisites for admission this present year (1860):—

1. The French language. 2. Arithmetic. 3. Geometry, plane and solid. 4. Algebra, to the general theory of equations. 5. Plane Trigonometry. 6. Analytical Geometry. 7. The elements of Descriptive Geometry. 8. General notions of Physics. 9. Inorganic Chemistry. 10. Natural History. 11. The elements of Anatomy and Physiology, Zoölogy and Botany. 12. Design, in which each student will present a collection of drawings relative to Descriptive Geometry and the tracing of curves of the second degree; also, some Architectural drawings shaded in India ink, and also a book of sketches of machines.

The applicants for admission are examined in writing, during which they have to make a drawing in Descriptive Geometry, and also one in Architecture, after a model. They are also afterward examined orally. Some idea of the labor and thoroughness of the examination will be formed from the fact that last year eight hundred persons were examined for admission, and of these only two hundred and fifty were admitted, the examination commencing the first of August and terminating the fifteenth of October.

The courses of the school commence the first of November, and last, including the examinations, until the middle of August. The students do not lodge in the establishment, but go every day before 8 ½ A. M., and remain until 4 P. M. All this time, except one hour for breakfast, is devoted to work of some kind, and during this time they are under the direction of the professors and inspectors.

There are attached to the school, sixteen professors, eleven repetiteurs, and two inspectors.

The duty of the professor is to give the instruction by lectures entirely, excepting such as must necessarily be given to each pupil separately, as in drawing and constructions.

The duty of each repetiteur is to examine and mark the students upon the subjects of the lectures of the professors, each student is examined in some course once every week, and *alone*, at which time he is obliged to show to the repetiteur the notes taken upon the subjects upon which he is examined.

The courses of Lectures for the first year are:—

1. Descriptive Geometry. 2. Physics. 3. Mechanics. 4. Chemistry. 5. Natural History. 6. Cinematics, or (Transformation of movements in machines).

The course of Descriptive Geometry comprehends the two treatises of Leroy on Descriptive Geometry and Sterotomie, in its Applications to Shades and Shadows, Linear Perspective, Dialing, Stone-cutting, and Carpentry, and besides the drawings of all of these, the students are required to make models of the different arches and domes, from chalk, and models in wood of the various details in Carpentry.

The students have two lectures in each course per week.

In the course of Physics, the students have such problems as this to solve. Determine the specific gravity of an alloy, taking into account the temperature of the water, and the barometric pressure.

In Chemistry each student makes twenty manipulations, of three hours each, from the preparation of oxygen to the preparation of the ethers.

The course on Mechanics commences with Analytic Geometry, and includes the Elements of the Differential and Integral Calculus, and also Theoretical Mechanics.

The students have problems to work. Here is one in Mechanics. A material point moves in a plane, and the laws of the velocities along two axes at right angles are known.

Deduce 1st, the co-ordinates of the moving point in functions of the time. 2d, Determine its

velocity. 3d, Calculate the acceleration along each axis, and thence the total, and also the tangential and normal acceleration. 4th, Discuss the curve in which the body moves, when

$$V_x = A\,(1 + e^{-w\,t}), \qquad A = 1, \qquad w = 0.004.$$
$$V_y = (a + b\,t), \qquad a = 2.4, \qquad b = 0.12.$$

The two courses of Natural History, and that of Cinematics, last five months each.

During vacation the students are required to make two drawings, one Architectural, and one of a machine; these are left to the selection of the student.

At the beginning of the second year each student chooses one speciality; these are four in number, viz. that of a Metallurgist, Constructor, Chemist, and Machinist. They have to follow all the courses but only execute projects and manipulations in their own speciality.

The courses are the same for the second and third years, except that a course upon Railroads is added the third year; they are as follows: — 1. Applied Mechanics. 2. Industrial Physics. 3. Industrial Chemistry. 4. Metallurgy. 5. *Exploitation* of Mines. 6. Analytical Chemistry. 7. Construction. 8. Construction of Machines.

Here is a project upon Applied Physics. To make the plans with all the details for the construction of a steam-boiler, which shall produce 700 kilogrammes of steam per hour.

Here is a project in the Construction of Machines. Construct a steam-engine to raise by means of pumps, the water to feed a reservoir at a railway station. The dimensions of the reservoir are 200 m. cube, the bottom of the reservoir is 6 m. above the ground. A noncondenser is required which shall work one or many pumps; the level of the water is 15 m. below the ground, and the amount required per hour is 33 m. cube.

In Construction projects are given, as the construction of a truss bridge of given dimensions. Or the construction of a viaduct in masonry. These projects are prepared in detail, and are often the work of a month or more of hard labor, consisting, as they do, of five or six large drawings and many pages of memoirs.

At the end, after a severe examination, if the student has passed successfully, he receives the diploma of civil engineer, without which he cannot practise in this country.

Discipline. Every student is obliged to be under the care of his parents or their correspondent, so that if there is any misconduct on the part of the student, his correspondent is immediately notified. Whenever the student is absent from school the correspondent is notified of that fact the same day. There are private and public reprimands given by the director, and in case the pupil is refractory, the correspondent is invited to remove him. I have been thus minute in describing the school for two reasons, that those young men who intend to finish their engineering education may know what sort of a school it is, and what they have to submit to, and also, that teachers may see that in the amount to be taught, if not in the methods, there is room for some improvement.

This is not by any means considered as equal in the theoretical instruction to the Polytechnic School, and its graduates cannot have that complete, and special, and practical skill, which the engineers trained in the Polytechnic, and afterward in the School of Mines or the École des Ponts et Chaussées. Of each one of these I shall have occasion to speak in detail.

I must not omit to say that the price of instruction in the Central School is 775 francs per year, and that the government has established a number of scholarships which are given to meritorious students; also, that the students, accompanied by their Professor of Design, make excursions to some edifice, or bridge, or manufactory in Paris, and make sketches and measurements of them on the spot, and afterwards draw them in detail. They also take excursions to make geological exploration in the neighborhood of Paris.

PARIS, August 6, 1860.

Editorial Items.

We have received the following solutions of the Prize Problems in the June number of the Monthly: —

Harriet S. Hazeltine, Worcester, Mass., Probs. I., II.

George B. Hicks, Cleveland, Ohio, Prob. III.

David Mercer, Okeana, Ohio, Probs. I., II.

W. T. Jackson, Western College, Linn Co., Iowa, Prob. II.

Asher B. Evans, Madison University, N. Y., Probs. I., II., III., IV., V.

O. B. Wheeler, Michigan University, Ann Arbor, Probs. III., IV., V.

Joseph B. Davison, Oberlin College, Ohio, Probs. I., II., III.

Gustavus Frankenstein, Springfield, Ohio, Prob. IV.

We have delayed the report of the Judges until our next number, in order to complete the article "On the Indeterminate Analysis" in this volume.

Prize Essays. — We have received four Prize Essays, and regret that Prof. Ferrel's absence has prevented the committee from giving their report in this number; but we hope to be able to announce the decision in our next.

Errata. — Page 108, for "$b^2 - d^2$" or "$b^2 - a^2$," read "$d^2 - b^2$"; page 181, line 16, "$x - y$" should precede "x"; page 329, Prob. III., for "tan 245°," read "tan 254°"; page 374, for "$t = 90°$," read "$t = -90°$."

Indexes. — Our special thanks are due to our friend and colleague, E. J. Loomis, Esq., for the Indexes to this volume.

The Illustrated Pilgrim Almanac for 1861. Published in Aid of the Monument Fund. Boston: Published at the Office of the National Monument to the Forefathers, No. 13 Tremont Row. Willard M. Harding, Financial Agent.

Most of our readers are probably already aware that the first number of this Almanac was for the year 1860, and published in August, 1859. The success of the first number was such as to warrant its continuance, and to demonstrate that the publication will largely aid the Monument Fund, not only directly, but by increasing public interest in the enterprise. It will be praise enough for the illustrations in the numbers before us, to say that they are by Hammatt Billings, Esq., the designer of the Pilgrim Monument.

The heads of the calendar pages in the number for 1860 represent several of the celebrated monuments of ancient and modern times; for the year 1861 they represent the discoveries of America, Florida, the Mississippi, St. Lawrence, Expedition of Sir Walter Raleigh, First Settlement of Virginia, Discovery of Hudson River, Settlement of Rhode Island, First Sabbath in New Haven, Settlement of Pennsylvania and Kentucky, and Emigration in the United States. Besides these, we find the Breakwater at Plymouth, England, Attempts of the Pilgrims to escape to Holland, the Pilgrim Meeting-House on Burying Hill, Plymouth, Capture of Annawan, and Immigration into the United States. The number closes with the names of 3,412 members of the Pilgrim Society who have subscribed $5, or upwards.

The calendars are given for five different parallels of latitude, — namely, Montreal, Boston, New York City, Washington, and Charleston, — thus making the work available for all parts of the United States and Canadas.

The work is beautifully printed on fine tinted paper, and is as ornamental as useful; and, at the low price of twenty-five cents a number, must have a large and steadily increasing sale.

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TEXT-BOOKS USED IN ACADEMIES IN THE STATE IN 1859.

Out of the 184 Academies in which Reading is taught, 114 used SANDERS'S, being more than three times the number of any other Readers.

Thomson's Arithmetics were used in eighty-three Academies, and far more than any other.

Robinson's Series, lately published, will also appear largely in next report.

Robinson's Algebra was already in eighty-eight Academies in 1859, and is fast gaining.

Colton's Geographies were in forty-seven Academies.

Gray's Botanies were in thirty-five Academies.

Fasquelle's French Course was in 133 out of the 167 Academies which teach French. **Fasquelle's Reading Books** are also gaining. As no other French work on the list exceeds twenty-three, Fasquelle was in more than seven times as many Academies as any other.

Woodbury's German Method was used in fifty-eight Academies, being in more than half the Academies that teach German, and nearly four times the number of any other. The next highest on the list is sixteen.

Wells's Natural Philosophy was in fifty-four Academies.

Wells's Chemistry was used in twenty-one Academies. These two books being but lately published, had not time to become extensively known until quite lately. They will be the leading books in the report of 1860.

Hitchcock's Geology was used in forty-seven Academies, — about four times the number of any other. It would have been still more largely in use but that many were waiting for the new edition just published.

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One Boston bookseller has sold *one hundred and nineteen* copies of Webster's Pictorial, and only *two* copies of Worcester, and states that "it is all that have been called for."

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